

Symmetry and uniqueness of minimizers of Hartree type equations with external Coulomb potential

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Abstract

In the present article we study the radial symmetry of minimizers of the energy functional, corresponding to the repulsive Hartree equation in external Coulomb potential. To overcome the difficulties, resulting from the "bad" sign of the nonlocal term, we modify the reflection method and then, by using Pohozaev integral identities we get the symmetry result.

Keywords: Hartree equations, minimizers, symmetry, variational methods, nonlinear solitary waves

2000 MSC: 35J50, 35J60, 35Q55

1. Introduction

Solitary waves associated with the Hartree type equation in external Coulomb potential are solutions of type

$$\chi(x)e^{-i\omega t}, \quad x \in \mathbb{R}^3, t \in \mathbb{R},$$

¹The first author was supported by the Italian National Council of Scientific Research (project PRIN No. 2008BLM8BB) entitled: "Analisi nello spazio delle fasi per E.D.P."

where $\omega > 0$ and χ satisfies the nonlinear elliptic equation

$$-\Delta\chi(x) + \int_{\mathbb{R}^3} \frac{|\chi(y)|^2 dy}{|x-y|} \chi(x) - \frac{\chi(x)}{|x|} + \omega\chi(x) = 0. \quad (1)$$

The natural energy functional associated with this problem is (see [4])

$$\mathcal{E}(\chi) = \frac{1}{2}\|\nabla\chi\|_{L^2}^2 + \frac{1}{4}A(|\chi|^2) - \frac{1}{2}\int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx, \quad (2)$$

where we shall denote

$$A(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} dy dx. \quad (3)$$

The corresponding minimization problem is associated with the quantity

$$I_N = \min\{\mathcal{E}(\chi); \chi \in H^1, \|\chi\|_{L^2}^2 = N\}. \quad (4)$$

The existence of positive minimizers $\chi_0(x)$, such that

$$\mathcal{E}(\chi_0) = I_N, \quad \|\chi_0\|_{L^2}^2 = N,$$

is established by Cazenave and Lions in [4] by the aid of the concentration compactness method.

For a given $\omega > 0$, the constrained minimization problem (4) can be compared with the unconstrained minimization problem

$$S_\omega^{min} = \min\{S_\omega(\chi); \chi \in H^1\},$$

where $S_\omega(\chi)$ is the corresponding action functional, defined by

$$S_\omega(\chi) = \mathcal{E}(\chi) + \frac{\omega}{2}\|\chi\|_{L^2}^2. \quad (5)$$

There are different results on the symmetry (and uniqueness) of the minimizers. The basic result due to Gidas, Ni and Nirenberg [9] implies the radial symmetry of the minimizers associated with the semilinear elliptic equation

$$\Delta u + f(u) = 0,$$

provided suitable assumptions on the function $f(u)$ are satisfied and the scalar function u is positive. As in the previous result due to Serrin [19], the proof is based on the maximum principle and the Hopf's lemma.

Therefore, the first natural question is to ask if the linear operator

$$P_\omega = -\Delta - \frac{1}{|x|} + \omega$$

in (1), satisfies the weak maximum principle in the sense that

$$u \in H^2, P_\omega(u) = g \geq 0, \implies u \geq 0. \quad (6)$$

The above maximum principle is incomplete, since additional behavior of u and g at infinity has to be imposed, namely, we shall suppose that

$$(1 + |x|)^{-M} e^{\sqrt{\omega}|x|} u \in H^2, \quad (1 + |x|)^{-M} e^{\sqrt{\omega}|x|} g \in H^2, \quad (7)$$

for some real number $M > 0$.

Note, that the energy levels of the hydrogen atom are described by the eigenvalues $\omega_k > 0$ of the eigenvalue problem

$$\Delta e_k(x) + \frac{e_k(x)}{|x|} = \omega_k e_k(x), \quad e_k(x) \in H^2.$$

One has

$$\omega_k = \frac{1}{4(k+1)^2}, \quad k = 0, 1, \dots$$

and $e_0(x) = ce^{-|x|/2}$, $c > 0$. The first observation is that all eigenfunctions $e_k(x)$, $k \geq 1$, are expressed in terms of Laguerre polynomials of $|x|$, having exactly k roots. This fact guarantees that the maximum principle is not valid for $\omega = \omega_k$. More precisely, we can show the following.

Lemma 1. *The weak maximum principle (6) is valid if and only if*

$$\omega \geq \frac{1}{4}.$$

This result can be compared with the existence of action minimizers for the corresponding functional S_ω , obtained by Lions for $0 < \omega < 1/4$ (see for details [14]).

Theorem 2. *We have the properties:*

a) *for any $\omega > 0$, the inequality*

$$\min_{\chi \in H^1} S_\omega(\chi) = S_\omega^{min} > -\infty$$

holds;

b) if $0 < \omega < 1/4$, then $S_\omega^{min} < 0$;

c) if $0 < \omega < 1/4$, then there exists a positive function $\chi(x) \in H^1$, such that

$$S_\omega(\chi) = S_\omega^{min}.$$

Our main goal of this paper is to clarify if the positive minimizers of S_ω are radially symmetric and unique. The above results show that we have to consider the domain $0 < \omega < 1/4$, where the key tool of Gidas, Ni and Nirenberg (i.e. the maximum principle for the corresponding linear operator) is not applicable.

The symmetry of the energy functional (even with constraint conditions) can not imply, in general, the radial symmetry of the minimizers. This phenomena was discovered and studied in the works [6], [7] and [8] in the scalar case.

Some sufficient conditions that guarantee the symmetry of minimizers have been studied by Lopes in [15], by means of the reflection method that (for the case of plane $x_1 = 0$) uses the functions

$$u_1(x) = \begin{cases} u(\hat{x}), & \hat{x} = (-x_1, x_2, \dots, x_n), \text{ if } x_1 > 0; \\ u(x), & \text{if } x_1 < 0 \end{cases}$$

and

$$u_2(x) = \begin{cases} u(\hat{x}), & \hat{x} = (-x_1, x_2, \dots, x_n), \text{ if } x_1 < 0; \\ u(x), & \text{if } x_1 > 0. \end{cases}$$

If the functional to be minimized has the form

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^n} F(u(x)) \, dx,$$

then we have the relation

$$E(u_1) + E(u_2) = 2E(u)$$

and this enables one to obtain the symmetry of minimizer, when $F(u)$ is a combination of functions of type $|u|^p, p \geq 2$.

The reflection method works effectively when $u(x)$ is a vector-valued function and constraint conditions (as in the problem (4)) are involved too.

Recently, the reflection method was generalized in [16] and [17] for very general situations and one example of possible application is the functional of type

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^n} F(u(x)) \, dx - A(|u|^2),$$

involving nonlocal term as in (2). This Choquard type functional has the specific property

$$E(u_1) + E(u_2) \leq 2E(u),$$

exploiting the negative sign of the nonlocal term $A(|u|^2)$.

An analogous result for the scalar case can be obtained by means of the Schwarz symmetrization (or spherical decreasing rearrangement [12]) $u^*(|x|)$ of the non-negative $u \in H^1$. Indeed, we have the equality

$$\int_{\mathbb{R}^n} F(u(x)) \, dx = \int_{\mathbb{R}^n} F(u^*(x)) \, dx,$$

as well as the inequalities

$$\|\nabla u\|_{L^2}^2 \geq \|\nabla u^*\|_{L^2}^2, \quad A(|u|^2) \leq A(|u^*|^2),$$

so, we get

$$E(u^*) \leq E(u)$$

and one can use the property that u is minimizer.

The functional in (2) is a typical example, when reflection method and Schwarz symmetrization meet essential difficulty to be applied directly.

The main goal of this work is to find an approach to establish the symmetry of the minimizer for functionals of Hartree type (2), involving nonlocal terms with "bad" sign.

To state this main result, we shall try first to connect the minimizers of the constraint problem (4) (associated with the energy functional $\mathcal{E}(\chi)$) with the minimization of the action functional $S_\omega(\chi)$. Similar relation for local type interactions is discussed in chapter IX of [3]. Then, we shall establish that the minimizer of Theorem 2 is a radially symmetric function.

Theorem 3. *The solution $\chi(x)$ from Theorem 2 is a radially symmetric function for*

$$\frac{1}{16} < \omega < \frac{1}{4}.$$

Remark 1. The result of Theorem III.1 in [4] treats more general case of potentials of type

$$V(x) = - \sum_{j=1}^K \frac{Z}{|x - x_j|},$$

while in our case we have

$$V(x) = - \frac{Z}{|x|}.$$

Therefore, the energy functional $\mathcal{E}(\chi)$ is rotationally invariant in our case. From Theorem 2 and Theorem 3 one can see that the solution $\chi_0(x)$ of (4) is radially symmetric and unique (up to a multiplication with complex number z , with $|z| = 1$).

As it was mentioned above the energy (and therefore the action) is a functional involving the nonlocal term with "bad" sign. To explain the main idea to treat this case, we recall the rotational symmetry of the energy (and action) functional. Therefore, if χ is the action minimizer from Theorem 2, it is sufficient to show that the solution is symmetric with respect to x_1 -plane, for any choice of the x_1 -direction. In other words, we consider $\hat{\chi}(x) = \chi(\hat{x})$, with $\hat{x} = (-x_1, x_2, x_3)$ and we aim to prove that $\chi = \hat{\chi}$.

To show this, we shall consider the two terms

$$\chi_{\pm} = \frac{\chi \pm \hat{\chi}}{2}.$$

So, our goal is to verify the inequality

$$S_{\omega}(\chi_+) + S_{\omega}(\chi_-) \leq S_{\omega}(\chi) \tag{8}$$

and see that the condition $\chi \neq \hat{\chi}$ implies $S_{\omega}(\chi_-) > 0$.

The form of the functional S_{ω} suggests one, in order to verify (8), to use an appropriate version of the Clarkson inequality for the quadratic form $A(f)$. Namely, we can prove that the following inequality

$$A\left(\left(\frac{f+g}{2}\right)^2\right) + A\left(\left(\frac{f-g}{2}\right)^2\right) \leq \frac{A(f^2) + A(g^2)}{2}$$

holds true. Unfortunately, the usual Clarkson inequality in the form given above, is too rough to serve as a tool for proving (8). Therefore, we shall use

a refined version of Clarkson inequality (see Lemma 5 below) in the form

$$A\left(\left(\frac{f+g}{2}\right)^2\right) + A\left(\left(\frac{f-g}{2}\right)^2\right) \leq \frac{A(f^2) + A(g^2)}{8} + \frac{3\sqrt{A(f^2)A(g^2)}}{4}.$$

The final step is to treat the uniqueness of positive minimizers. of the problem

$$S_\omega^{min} = \min\{S_\omega(\chi); \chi \in H^1\}. \quad (9)$$

Our proof can not follow the Lieb's uniqueness proof for the ground state solution of the Choquard equation [11]. In general, the Lieb's proof strongly depends on the specific features of the nonlocal nonlinear equation (1) and differs from the corresponding results for semilinear elliptic equation given by Kwong in [10]. Indeed, once the radial symmetry is established, one can use Pohozaev identities and reduce the nonlocal nonlinear elliptic problem (1) to an ordinary differential equation of the type

$$u''(r) + W(r)u(r) + 4\pi \int_0^r \left(\frac{1}{s} - \frac{1}{r}\right) u^2(s) ds u(r) = \omega u(r),$$

where

$$W_\chi(r) = \frac{1}{r} - 4\pi \int_0^\infty \chi^2(s) s ds.$$

The positive sign in front of the nonlinear term is the main obstacle to apply Sturm type argument and derive the uniqueness of positive solutions to this ordinary differential equation. However, for $\frac{1}{16} < \omega < \frac{1}{4}$ we can apply the approach based on the refined Clarkson inequality and using the orthogonal projection on the eigenspace of the first eigenvalue of the operator $\Delta + 1/|x|$, we can establish the following result.

Theorem 4. *Let $\frac{1}{16} < \omega < \frac{1}{4}$. Then, the solution χ of minimization problems (9) is unique.*

Let's mention that the results in Theorems 3 and 4 can be compared with the results in [1], where the uniqueness of minimizers for the constrained variational problem (4) is studied. To show the relations between action minimization and (4) one has to apply the uniqueness of action minimizers or alternatively the uniqueness of minimizers of constrained variational problem.

The plan of the work is the following. In Section 2 we consider the maximum principle for the linear Schrödinger equation with Coulomb potential and prove Lemma 1. The proof of Theorem 3, stating that the minimizers are radially symmetric is presented in Section 3 by the aid of a refined version of Clarkson inequality. In Section 4 we establish the Pohozaev integral relations, corresponding to equation (1), and in Section 5 we prove uniqueness Theorem 4. Finally, in Appendix A we prove for completeness the existence of positive action minimizers, stated in Theorem 2, while in Appendix B the connection between energy and action minimizers is discussed.

The authors are grateful to Louis Jeanjean for important discussions and remarks on symmetry of minimizers as well as to the referee for pointing out a gap in the proof of the Theorem 3.

2. Maximum principle for Schrödinger equation with Coulomb potential

The maximum principle, stated in (6) will be verified by the aid of the substitution

$$u = \varphi w, \varphi(x) = \varphi(|x|),$$

where φ is a radial function, satisfying the property

$$-\Delta\varphi - \frac{\varphi}{|x|} + \omega\varphi = h(|x|) \geq 0. \quad (10)$$

Our goal is to construct φ , so that $\varphi(|x|) > 0$. We have several possibilities, depending on ω . If $\omega > 1/4$, we shall show that such a function exists and it is of type

$$\varphi(r) = e^{-\beta r} Q(r), \quad \beta = \sqrt{\omega}, \quad Q(r) = Ar^2 + Br + C. \quad (11)$$

If $\omega = 1/4$, then we can take simply $\varphi(r) = e^{-r/2}$. If $0 < \omega < 1/4$, we shall see that a function φ of type (11) exists, but $\varphi(r)$ changes the sign for $r > 0$. Hence, this function gives a counterexample, showing that the weak maximum principle (6) is not fulfilled in this case.

Therefore, to complete the proof of Lemma 1, we have to explain how the existence of positive $\varphi(r)$, satisfying (10) will imply the weak maximum principle and then to construct in different cases the function $Q(r)$ in (11), so that (10) is satisfied.

PROOF OF LEMMA 1. After the substitution $u = \varphi w$, we have

$$P_\omega(u) = -\varphi\Delta w - 2\nabla\varphi\nabla w + P_\omega(\varphi)w = \varphi\Delta w + 2\nabla\varphi\nabla w + hw.$$

If $\varphi(|x|) > 0$, then we can write

$$-\Delta w - \frac{2}{\varphi}\nabla\varphi\nabla w + \frac{h}{\varphi}w = \frac{g}{\varphi}.$$

Choosing $M = 1$, we see that

$$\frac{g}{\varphi} \in H^2,$$

so we can apply the classical maximum principle (since $h \geq 0$) and obtain $w \geq 0$. This argument shows that the maximum principle is fulfilled if the function $\varphi(r)$ satisfies inequality (10) and its polynomial term $Q(r) > 0$ for $r \geq 0$.

To construct Q , we substitute $\varphi(r) = e^{-\beta r}Q(r)$ into (10) and find that

$$e^{\beta r}rh(r) = -(2B + C(-2\beta + 1)) - (6A + B(-4\beta + 1))r + (6\beta - 1)Ar^2.$$

We take for simplicity $A = 1$ and

$$B = C(\beta - 1/2), \quad C = \frac{12}{(2\beta - 1)(4\beta - 1)}.$$

Then the condition $\beta > 1/2$ implies that

$$B = \frac{6}{(4\beta - 1)} > 0, \quad C = \frac{12}{(2\beta - 1)(4\beta - 1)} > 0$$

so $Q(r) > 0$ and

$$e^{\beta r}rh(r) = (6\beta - 1)r^2 \geq 0.$$

This argument completes the proof of the weak maximum principle for $\omega > 1/4$.

If $1/16 < \omega < 1/4$, then we can take the same A, B, C and see that

$$e^{\beta r}rh(r) = (6\beta - 1)r^2 \geq 0.$$

Since $A = 1$ and $C < 0$ in this case, the function $Q(r)$ changes the sign.

Finally, if $0 < \omega < 1/16$, then we choose

$$A = 0, B = -1, C = \frac{1}{1/2 - \beta}$$

and then

$$Q(r) = \frac{2}{1 - 2\beta} - r, \quad e^{\beta r} r h(r) = (1 - 4\beta)r \geq 0.$$

Again, it is clear that $Q(r)$ changes the sign, and the proof of the Lemma is completed.

3. Radial symmetry of action minimizers

Even in the non-local case, the problem that action and energy minimizers are nonnegative functions, is easy to be proved. Indeed, if $\chi(x) \in H^1$ is a real-valued minimizer of the functional

$$S_\omega(\chi) = \frac{1}{2} \|\nabla \chi\|_{L^2}^2 + \frac{1}{4} A(\chi^2) - \frac{1}{2} \int_{\mathbb{R}^3} \frac{\chi(x)^2}{|x|} dx + \frac{\omega}{2} \|\chi\|_{L^2}^2, \quad (12)$$

then $|\chi(x)|$ satisfies the inequality

$$\|\nabla |\chi|\|_{L^2}^2 \leq \|\nabla \chi\|_{L^2}^2,$$

as well as the identities

$$A(|\chi|^2) = A(\chi^2), \quad \int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx = \int_{\mathbb{R}^3} \frac{\chi(x)^2}{|x|} dx,$$

so $|\chi(x)| \geq 0$ is also a minimizer of S_ω .

Let us define the bilinear form

$$L_\omega(\chi, \psi) = \langle (-\Delta - \frac{1}{|x|} + \omega)\chi, \psi \rangle_{L^2}, \quad \omega > 0 \quad (13)$$

and the corresponding quadratic form

$$L_\omega(\chi) = \langle (-\Delta - \frac{1}{|x|} + \omega)\chi, \chi \rangle_{L^2}. \quad (14)$$

The quadratic form $A(\chi)$ defined in (3) generates the corresponding bilinear form

$$A(\chi, \psi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\chi(x)\psi(y)}{|x-y|} dy dx. \quad (15)$$

Then, the action functional S_ω can be written as

$$S_\omega(\chi) = \frac{1}{2}L_\omega(\chi) + \frac{1}{4}A(\chi^2). \quad (16)$$

Also, for any function χ we shall denote $\hat{\chi}(x) = \chi(\hat{x})$, where $\hat{x} = (-x_1, x_2, x_3)$ for any choice of our x_1 -axis. It is easy to check that

$$S_\omega(\chi) = S_\omega(\hat{\chi}), \quad L_\omega(\chi) = L_\omega(\hat{\chi}). \quad (17)$$

With our next result, we shall establish Clarkson type inequalities for the forms A and L_ω . In fact, we shall prove the Lemma.

Lemma 5. *The following inequalities hold*

$$L_\omega\left(\frac{f+g}{2}\right) + L_\omega\left(\frac{f-g}{2}\right) = \frac{L_\omega(f) + L_\omega(g)}{2}, \quad (18)$$

$$\begin{aligned} A\left(\left(\frac{f+g}{2}\right)^2\right) + A\left(\left(\frac{f-g}{2}\right)^2\right) &\leq \frac{A(f^2) + A(g^2)}{8} \\ &\quad + \frac{3\sqrt{A(f^2)A(g^2)}}{4}. \end{aligned} \quad (19)$$

PROOF. It is easy to verify the relation

$$\begin{aligned} &A\left(\left(\frac{f+g}{2}\right)^2\right) + A\left(\left(\frac{f-g}{2}\right)^2\right) \\ &= \frac{1}{16}A(f^2 + g^2 + 2fg) + \frac{1}{16}A(f^2 + g^2 - 2fg). \end{aligned} \quad (20)$$

Note that from

$$A(a+b) + A(a-b) = 2A(a) + 2A(b),$$

equality (20) becomes

$$\begin{aligned} A\left(\left(\frac{f+g}{2}\right)^2\right) + A\left(\left(\frac{f-g}{2}\right)^2\right) &= \frac{1}{8} [A(f^2 + g^2) + 4A((fg)^2)] \\ &= \frac{1}{8} [A(f^2) + A(g^2) + 2A(f^2, g^2) + 4A((fg)^2)] \\ &\leq \frac{A(f^2) + A(g^2)}{8} + \frac{3\sqrt{A(f^2)A(g^2)}}{4}, \end{aligned} \quad (21)$$

which proves (19). The first relation (18) in the Lemma, follows directly.

The next result will play the crucial role in the present study. We shall prove the following Lemma.

Lemma 6. *If $L_\omega(f) = L_\omega(g)$ and $\mu, \nu \geq 0$ satisfy $2(\mu^2 + \nu^2) = 1$, then*

$$L_\omega(\mu f + \nu g) + L_\omega(\mu f - \nu g) = L_\omega(f). \quad (22)$$

If $A(f^2) = A(g^2)$ and $\mu, \nu \geq 0$ satisfy $2(\mu^2 + \nu^2) = 1$, then we have

$$A((\mu f + \nu g)^2) + A((\mu f - \nu g)^2) \leq A(f^2). \quad (23)$$

PROOF. Setting $\mu_1 = 2\mu$, $\nu_1 = 2\nu$, we apply (18) with f, g replaced by $\mu_1 f$ and $\nu_1 g$ respectively. Thus, we get

$$L_\omega\left(\frac{\mu_1 f + \nu_1 g}{2}\right) + L_\omega\left(\frac{\mu_1 f - \nu_1 g}{2}\right) = \frac{\mu_1^2 L_\omega(f) + \nu_1^2 L_\omega(g)}{2}. \quad (24)$$

From $L_\omega(f) = L_\omega(g)$ and $\mu_1^2 + \nu_1^2 = 2$, we complete the proof of (22).

Similarly, applying (19) and the assumption $A(f^2) = A(g^2)$, we find

$$A\left(\left(\frac{\mu_1 f + \nu_1 g}{2}\right)^2\right) + A\left(\left(\frac{\mu_1 f - \nu_1 g}{2}\right)^2\right) \leq \frac{\mu_1^4 + \nu_1^4 + 6\mu_1^2 \nu_1^2}{8} A(f^2)$$

or, equivalently

$$A\left(\left(\frac{\mu f + \nu g}{2}\right)^2\right) + A\left(\left(\frac{\mu f - \nu g}{2}\right)^2\right) \leq 2(\mu^4 + \nu^4 + 6\mu^2 \nu^2) A(f^2) \quad (25)$$

Consider now the homogeneous quartic polynomial

$$2(\mu^4 + \nu^4 + 6\mu^2 \nu^2) \quad (26)$$

on the circle $\mu^2 + \nu^2 = \frac{1}{2}$. Substituting $\nu^2 = \frac{1}{2} - \mu^2$, we obtain the following estimate

$$\begin{aligned} 2(\mu^4 + \nu^4 + 6\mu^2 \nu^2) &= 2((\mu^2 + \nu^2)^2 + 4\mu^2 \nu^2) \\ &= \frac{1}{2} + 4\mu^2 - 8\mu^4 = 1 - \frac{(1 - 4\mu^2)^2}{2} \leq 1. \end{aligned} \quad (27)$$

Then, from (25) and (27) follows the proof of the Lemma.

Turning back to the minimization problem of the action functional S_ω , we observe the following fact. If $\chi(x)$ is a minimizer of the problem

$$\min_{\chi \in H^1} S_\omega(\chi), \quad (28)$$

then $\hat{\chi}(x)$ and $-\hat{\chi}(x)$ are also minimizers of $S_\omega(\chi)$. Moreover, we have the property.

Lemma 7. *Assume that $\chi(x)$ is a minimizer of the problem (28) and one of the following alternatives:*

1. $L_\omega(\chi - \hat{\chi}) \geq 0$;
2. $L_\omega(\chi + \hat{\chi}) \geq 0$

holds. Then $\chi = \hat{\chi}$.

PROOF. For simplicity, we shall consider the first case only. Suppose $\chi \neq \hat{\chi}$, then from (18) we have

$$L_\omega\left(\frac{\chi + \hat{\chi}}{2}\right) + L_\omega\left(\frac{\chi - \hat{\chi}}{2}\right) = L_\omega(\chi), \quad (29)$$

implying

$$L_\omega\left(\frac{\chi + \hat{\chi}}{2}\right) \leq L_\omega(\chi). \quad (30)$$

On the other hand, it is easy to check that the following Cauchy inequalities

$$A(f^2, g^2) \leq \sqrt{A(f^2)A(g^2)}, \quad A(fg) \leq \sqrt{A(f^2)A(g^2)} \quad (31)$$

hold true. Applying now (19), we obtain

$$\begin{aligned} A\left(\left(\frac{\chi + \hat{\chi}}{2}\right)^2\right) + A\left(\left(\frac{\chi - \hat{\chi}}{2}\right)^2\right) &\leq \frac{A(\chi^2) + A(\hat{\chi}^2)}{8} \\ &+ \frac{3\sqrt{A(\chi^2)A(\hat{\chi}^2)}}{4} \leq \frac{A(\chi^2) + A(\hat{\chi}^2)}{2} = A(\chi^2), \end{aligned} \quad (32)$$

which, together with the assumption $\chi \neq \hat{\chi}$ gives that

$$A\left(\left(\frac{\chi + \hat{\chi}}{2}\right)^2\right) < A(\chi^2). \quad (33)$$

Thus, from (30), (33) and the definition (16) it follows

$$S_\omega \left(\frac{\chi + \hat{\chi}}{2} \right) \leq S_\omega(\chi), \quad (34)$$

which contradicts to the assumption that χ is a minimizer. This proves the Lemma.

Now, we are ready to prove the radial symmetry of the action minimizer, stated in Theorem 3.

PROOF OF THEOREM 3. Taking into account Lemma 7, we shall take a minimizer $\chi(x) \geq 0$ of S_ω and shall show that the condition

$$\frac{1}{16} < \omega < \frac{1}{4},$$

implies that $\chi = \hat{\chi}$ or

$$L_\omega(\chi - \hat{\chi}) > 0. \quad (35)$$

Let

$$\chi(x) = e_0(x) + f(x),$$

where $e_0(x) = ce^{-|x|/2}$, $c > 0$ is the eigenvector corresponding to the first eigenvalue of the operator $\Delta + 1/|x|$, while $\langle f, e_0 \rangle_{L^2} = 0$. Since e_0 is a radial function, we have $\hat{e}_0 = e_0$, so

$$\chi - \hat{\chi} = f - \hat{f} = g, \quad \langle g, e_0 \rangle_{L^2} = 0, \quad g \neq 0.$$

Lemma 8. *Let us assume that $g \perp e_0$ in L^2 . Then*

$$L_\omega(g) \geq \left(\omega - \frac{1}{16} \right) \|g\|_{L^2}^2.$$

PROOF. Note that $g \perp e_0$ in L^2 implies

$$g = \sum_{k \geq 1} c_k e_k + h,$$

where h is in the absolutely continuous space of the self-adjoint operator $\Delta + \frac{1}{|x|}$ in L^2 , while e_k are eigenvectors of the same operator in $\{g \in L^2; g \perp$

$e_0\}$ with eigenvalues $\omega_k \leq 1/16$. On the absolutely continuous space the operator has spectrum on $(-\infty, 0)$ and it is non positive, so

$$\left\langle \left(\Delta + \frac{1}{|x|} \right) h, h \right\rangle \leq 0.$$

Hence, we have

$$\left\langle \left(\Delta + \frac{1}{|x|} \right) g, g \right\rangle \leq \sum |c_k|^2 \omega_k \leq \frac{1}{16} \left(\sum |c_k|^2 \right) = \frac{1}{16} \|g\|_{L^2}^2$$

and

$$L_\omega(g) = - \left\langle \left(\Delta + \frac{1}{|x|} \right) g, g \right\rangle + \omega \|g\|_{L^2}^2 \geq \left(\omega - \frac{1}{16} \right) \|g\|_{L^2}^2.$$

This completes the proof of the Lemma.

Applying the above Lemma, we find

$$L_\omega(\chi - \hat{\chi}) = L_\omega(g) \geq \left(\omega - \frac{1}{16} \right) \|g\|_{L^2}^2 > 0,$$

since $\omega > 1/16$ and $g \neq 0$. Hence, (35) is fulfilled and the proof of the Theorem is complete.

4. Pohozaev identities

In this part we shall establish the so-called Pohozaev identities for (1). More precisely, we shall prove the following

Lemma 9. *If $\chi \in H^1(\mathbb{R}^3)$ and satisfies (1) in $H^{-1}(\mathbb{R}^3)$, then the following identities hold*

$$\|\nabla \chi\|_{L^2}^2 + \omega \|\chi\|_{L^2}^2 = \int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx - A(|\chi|^2), \quad (36)$$

$$\|\nabla \chi\|_{L^2}^2 + 3\omega \|\chi\|_{L^2}^2 = 2 \int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx - \frac{5}{2} A(|\chi|^2). \quad (37)$$

PROOF. To prove (36) we multiply equation (1) by $\bar{\chi}$, take the real part and integrate over \mathbb{R}^3 . To prove (37) we shall use the following relations

$$\nabla \cdot (x|\chi|^2) = 3|\chi|^2 + 2 \operatorname{Re} \chi(x \cdot \nabla \bar{\chi}), \quad (38)$$

$$\nabla \cdot (x|\nabla \chi|^2 - 2 \operatorname{Re} \nabla \chi(x \cdot \nabla \bar{\chi})) = |\nabla \chi|^2 - 2 \operatorname{Re} \Delta \chi(x \cdot \nabla \bar{\chi}), \quad (39)$$

$$\nabla \cdot \left(x \frac{|\chi|^2}{|x|} \right) = 2 \frac{|\chi|^2}{|x|} + 2 \operatorname{Re} \frac{\chi(x \cdot \nabla \bar{\chi})}{|x|}, \quad (40)$$

and

$$\begin{aligned} \nabla \cdot \left(x \int_{\mathbb{R}^3} \frac{|\chi(y)|^2 dy}{|x-y|} |\chi|^2 \right) &= 3 \int_{\mathbb{R}^3} \frac{|\chi(y)|^2 dy}{|x-y|} |\chi|^2 \\ &- \int_{\mathbb{R}^3} \frac{x(x-y)|\chi(y)|^2 dy}{|x-y|^3} |\chi|^2 + 2 \int_{\mathbb{R}^3} \frac{|\chi(y)|^2 dy}{|x-y|} \operatorname{Re} \chi(x \cdot \nabla \bar{\chi}). \end{aligned} \quad (41)$$

Integrating (38)–(41) over \mathbb{R}^3 implies the equalities

$$\operatorname{Re} \int_{\mathbb{R}^3} \chi(x \cdot \nabla \bar{\chi}) \, dx = -\frac{3}{2} \|\chi\|_{L^2}^2, \quad (42)$$

$$\operatorname{Re} \int_{\mathbb{R}^3} \Delta \chi(x \cdot \nabla \bar{\chi}) \, dx = \frac{1}{2} \|\nabla \chi\|_{L^2}^2, \quad (43)$$

$$\operatorname{Re} \int_{\mathbb{R}^3} \frac{1}{|x|} \chi(x \cdot \nabla \bar{\chi}) \, dx = - \int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} \, dx, \quad (44)$$

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\chi(y)|^2 \chi(x)(x \cdot \nabla \bar{\chi}(x))}{|x-y|} dy dx &= -\frac{3}{2} A(|\chi|^2) \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x(x-y)|\chi(y)|^2 |\chi(x)|^2}{|x-y|^3} dy dx. \end{aligned} \quad (45)$$

On the other hand, observing the symmetry

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x(x-y)|\chi(y)|^2 |\chi(x)|^2}{|x-y|^3} dy dx \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{y(y-x)|\chi(y)|^2 |\chi(x)|^2}{|x-y|^3} dy dx, \end{aligned} \quad (46)$$

we calculate

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x(x-y)|\chi(y)|^2|\chi(x)|^2}{|x-y|^3} dydx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(x-y)^2|\chi(y)|^2|\chi(x)|^2}{|x-y|^3} dydx = \frac{1}{2}A(|\chi|^2). \end{aligned} \quad (47)$$

Substituting (47) into (45) we get

$$\operatorname{Re} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\chi(y)|^2\chi(x)(x \cdot \nabla \bar{\chi}(x))}{|x-y|} dydx = -\frac{5}{4}A(|\chi|^2). \quad (48)$$

Finally, multiplying equation (1) by $x \cdot \nabla \bar{\chi}$, taking the real part, integrating over \mathbb{R}^3 and using (42), (43), (44) and (48) we complete the proof of the Lemma.

The Pohozaev identities are useful to treat the uniqueness of the minimizers (modulo multiplication by complex constant z with $|z| = 1$). Indeed, let χ_1 and χ_2 are minimizers of the problem

$$S_\omega^{min} = \min\{S_\omega(\chi); \chi \in H^1\}. \quad (49)$$

Since

$$S_\omega(\chi) = \frac{1}{2}\|\nabla \chi\|_{L^2}^2 + \frac{\omega}{2}\|\chi\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx + \frac{1}{4}A(|\chi|^2),$$

we can apply the Pohozaev identities of Lemma 9. In this way we find

$$S_\omega(\chi) = -\frac{1}{4}A(|\chi|^2) \quad (50)$$

and

$$A(|\chi_1|^2) = A(|\chi_2|^2), \quad L_\omega(\chi_1) = L_\omega(\chi_2), \quad (51)$$

where $L_\omega(\chi)$ is defined according to (14).

5. Uniqueness of minimizers

In this section we shall prove the uniqueness result of Theorem 4. The classical approach for proving the uniqueness of minimizers is to reduce the

initial nonlinear equation to an ordinary differential equation, using the radial symmetry. Uniqueness of positive ground state solutions for nonlinear Schrödinger equation on \mathbb{R}^n with local nonlinearities of the form $|u|^p u$ for $0 < p < \frac{4}{n-2}$, is a well-known fact, due to Kwong [10]. The proof in this case relies on Sturm comparison theorems, but it cannot be applied directly to nonlocal equations, such as (1). For the attractive Choquard equation, Lieb in [11] prove uniqueness of energy minimizer by using Newton's theorem for radial function $f(x) = f(|x|)$, that is

$$\int \frac{f(|y|)}{|x-y|^{n-2}} dy = \int \frac{f(|y|)}{\max\{|x|, |y|\}} dy. \quad (52)$$

The repulsive sign of the Hartree term in (1) is again the main obstacle for applying directly the standard technique.

PROOF OF THEOREM 4. Let χ_1 and χ_2 are non negative minimizers of the problem

$$S_\omega^{min} = \min\{S_\omega(\chi); \chi \in H^1\}.$$

Since they are radial functions, one can rewrite the elliptic equation (1), using Newton's theorem (52), as an ordinary differential equation of the form

$$-\chi''(r) - \frac{2}{r}\chi'(r) - \frac{\chi(r)}{r} + 4\pi \int_0^\infty \frac{\chi^2(s)s^2 ds}{\max\{r, s\}} \chi(r) + \omega\chi(r) = 0. \quad (53)$$

The above equation can be rewritten in the form

$$-\chi''(r) - \frac{2}{r}\chi'(r) - W(r)\chi(r) + 4\pi \int_0^r \chi^2(s) \left(\frac{1}{r} - \frac{1}{s}\right) s^2 ds \chi(r) + \omega\chi(r) = 0,$$

where

$$W(r) = \frac{1}{r} - 4\pi \int_0^\infty \chi^2(s)s ds.$$

If we set $u(r) = r\chi(r)$, then from the identity

$$\chi''(r) + \frac{2}{r}\chi'(r) = \frac{u''(r)}{r},$$

the last equation becomes

$$u''(r) + W(r)u(r) - 4\pi \int_0^r \left(\frac{1}{r} - \frac{1}{s}\right) u^2(s) ds u(r) = \omega u(r). \quad (54)$$

This observation shows that the assumption $\chi(x)$ is a non negative minimizer implies $u(r) > 0$ for $r > 0$. Hence $\chi_1(x)$ and $\chi_2(x)$ are positive functions.

Our goal is to use the projection of χ_1 and χ_2 on the one dimensional eigenspace

$$E_0 = \{\alpha e^{-|x|/2}, \alpha \in (-\infty, \infty)\}$$

is the eigenvector corresponding to the first eigenvalue $\omega_0 = 1/4$ of the operator $\Delta + 1/|x|$. First, we have to observe that χ_1 is not orthogonal to E_0 . Indeed, if $\chi_1 \perp E_0$, then Lemma 8 implies

$$L_\omega(\chi_1) \geq \left(\omega - \frac{1}{16}\right) \|\chi_1\|_{L^2}^2 > 0.$$

The relation (16) guarantees now $S_\omega(\chi_1) > 0$ and this contradicts the relation (50). The contradiction shows that χ_1 (and also χ_2) is not orthogonal to E_0 .

Let

$$\chi_1 = \mu_1 \alpha e^{-|x|/2} + f_1, \quad \chi_2 = \mu_2 \alpha e^{-|x|/2} + f_2,$$

where $\alpha e^{-|x|/2} \in E_0$, with $\alpha > 0$ and $f_1, f_2 \perp E_0$. Note that $\mu_1, \mu_2 > 0$, since χ_1, χ_2 and e_0 are positive functions. We can choose $\alpha > 0$, such that

$$2(\mu_1^2 + \mu_2^2) = 1, \tag{55}$$

used as assumption in Lemma 6. The other assumption

$$A(|\chi_1|^2) = A(|\chi_2|^2), \quad L_\omega(\chi_1) = L_\omega(\chi_2),$$

is already established in (51).

Applying Lemma 6, we find the identity

$$L_\omega(\mu_2 \chi_1 + \mu_1 \chi_2) + L_\omega(\mu_2 \chi_1 - \mu_1 \chi_2) = L_\omega(\chi_1),$$

as well as the inequality

$$A((\mu_2 \chi_1 + \mu_1 \chi_2)^2) + A((\mu_2 \chi_1 - \mu_1 \chi_2)^2) \leq A(\chi_1^2).$$

Then, we have the relation

$$\mu_2 \chi_1 - \mu_1 \chi_2 = \mu_2 f_1 - \mu_1 f_2 = g \perp E_0.$$

If $g = 0$, then $\chi_1 = \mu_1 \chi_2 / \mu_2$ and one can use the ODE (54) and the corresponding integral identities (36) and (37), to show that $\chi_1 = \chi_2$. If $g \neq 0$, then one can apply Lemma 8 and find

$$L_\omega(\mu_2 \chi_1 - \mu_1 \chi_2) \geq \left(\omega - \frac{1}{16} \right) \|g\|_{L^2}^2 > 0.$$

Hence,

$$S(\mu_2 \chi_1 + \mu_1 \chi_2) = \frac{1}{2} L_\omega(\mu_2 \chi_1 + \mu_1 \chi_2) + \frac{1}{4} A((\mu_2 \chi_1 + \mu_1 \chi_2)^2) < S_\omega(\chi_1)$$

and this is a contradiction. The contradiction shows that $\chi_1 = \chi_2$ and this completes the proof of Theorem 4.

Appendix A. Existence of action minimizers

The existence of action minimizers for Hartree type equation is already established in [14]. For completeness, we shall sketch the proof.

To show the boundedness from below of S_ω , we shall prove the following inequalities involving homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s(\mathbb{R}^3)} = \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^3)}, \quad s > -3/2.$$

Lemma 10. *For any $p_1 \in [3, 6]$ and $p_2 \in [2, 3]$ we have the estimates*

$$\left(\int_{|x| \leq 1} |\chi(x)|^{p_1} dx \right)^{1/p_1} \leq C \|\chi\|_{\dot{H}^1}^{\theta_1} \|\chi^2\|_{\dot{H}^{-1}}^{(1-\theta_1)/2} \quad (\text{A.1})$$

$$\left(\int_{|x| \geq 1} |\chi(x)|^{p_2} dx \right)^{1/p_2} \leq C \|\chi\|_{L^2}^{\theta_2} \|\chi\|_{\dot{H}^1}^{\theta_3} \|\chi^2\|_{\dot{H}^{-1}}^{(1-\theta_2-\theta_3)/2}, \quad (\text{A.2})$$

where

$$\theta_1 = \frac{5}{3} - \frac{4}{p_1}, \quad \theta_2 = \frac{4(3-p_2)}{p_2}, \quad \theta_3 = \frac{p_2-2}{p_2}.$$

Remark 2. The assumptions $p_1 \in [3, 6]$ and $p_2 \in [2, 3]$ guarantee that all parameters $\theta_1, \theta_2, \theta_3, \theta_2 + \theta_3$ are in the interval $[0, 1]$.

Remark 3. The relation

$$\|f\|_{\dot{H}^{-1}}^2 = \langle (-\Delta)^{-1}f, f \rangle_{L^2} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} dy dx$$

implies

$$\|\chi^2\|_{\dot{H}^{-1}}^2 = \frac{1}{4\pi} A(\chi^2).$$

PROOF. For $p_1 = 6$ the inequality (A.1) becomes

$$\left(\int_{|x| \leq 1} |\chi(x)|^6 dx \right)^{1/6} \leq C \|\chi\|_{\dot{H}^1}$$

and this is the standard Sobolev embedding. For $p_1 = 3$ we have to verify the following estimate

$$\left(\int_{\mathbb{R}^3} |\chi(x)|^3 dx \right)^{1/3} \leq C \|\chi\|_{\dot{H}^1}^{1/3} \|\chi^2\|_{\dot{H}^{-1}}^{1/3}. \quad (\text{A.3})$$

This inequality follows from

$$\left| \int f(x)g(x)dx \right| \leq \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^{-1}}$$

with $f(x) = |\chi(x)|$, $g(x) = |\chi(x)|^2 = \chi^2(x)$ and the observation that

$$\||\chi|\|_{\dot{H}^1} = \|\chi\|_{\dot{H}^1}.$$

Interpolation between $p_1 = 6$ and $p_1 = 3$ proves (A.1).

The inequality (A.2) for $p_2 = 3$ follows from (A.3).

For $p_2 = 2$ (A.2) reduces to the simple inequality

$$\left(\int_{|x| \geq 1} |\chi(x)|^2 dx \right)^{1/2} \leq C \|\chi\|_{L^2}.$$

An interpolation argument implies (A.2) and completes the proof of the Lemma.

After this Lemma we can show that the action functional is bounded from below.

Lemma 11. *For any $\omega > 0$ the inequality*

$$\min_{\chi \in H^1} S_\omega(\chi) = S_\omega^{min} > -\infty$$

holds. For $0 < \omega < 1/4$ we have $S_\omega^{min} < 0$.

PROOF. The only negative term in S_ω is

$$-\frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx.$$

Decomposing the integration domain into $|x| \leq 1$ and $|x| > 1$ we apply Hölder inequality and obtain

$$\int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx \leq C \left(\int_{|x| \leq 1} |\chi(x)|^{p_1} dx \right)^{2/p_1} + C \left(\int_{|x| > 1} |\chi(x)|^{p_2} dx \right)^{2/p_2},$$

where $p_1 > 3 > p_2$. Applying Lemma 10 as well as the Young inequality

$$X^{\theta_1} Y^{\theta_2} Z^{\theta_3} \leq \varepsilon X + \varepsilon Y + C_\varepsilon Z,$$

with

$$\theta_j \in (0, 1), \theta_1 + \theta_2 + \theta_3 = 1,$$

we get

$$\int_{\mathbb{R}^3} \frac{|\chi(x)|^2}{|x|} dx \leq \varepsilon \|\chi\|_{L^2}^2 + \varepsilon \|\nabla \chi\|_{L^2}^2 + C_\varepsilon \sqrt{A(\chi^2)}.$$

This estimate implies

$$S_\omega(\chi) \geq \frac{1-\varepsilon}{2} \|\nabla \chi\|_{L^2}^2 + \frac{\omega-\varepsilon}{2} \|\chi\|_{L^2}^2 + \frac{1}{4} A(\chi^2) - C_\varepsilon \sqrt{A(\chi^2)}.$$

Choosing $\varepsilon > 0$ so small that $\varepsilon < \min(1, \omega)$, we find

$$S_\omega(\chi) \geq \frac{1}{4} A(\chi^2) - C_\varepsilon \sqrt{A(\chi^2)} \geq -2C_\varepsilon^2.$$

To finish the proof we take $\chi_\delta(x) = \delta e^{-|x|/2}$, such that

$$\left(\Delta + \frac{1}{|x|} \right) \chi_\delta = \frac{1}{4} \chi_\delta.$$

Then

$$2S_\omega(\chi_\delta) = (\omega - 1/4)\|\chi_\delta\|_{L^2}^2 + A(\chi_\delta^2)/2.$$

Since

$$\|\chi_\delta\|_{L^2}^2 = C_0\delta^2, \quad A(\chi_\delta^2)/2 = O(\delta^4),$$

the condition $\omega \in (0, 1/4)$ implies $2S_\omega(\chi_\delta) < 0$ and this completes the proof.

PROOF OF THEOREM 2. Take a minimizing sequence $\chi_k \in H^1$, so that

$$\lim_{k \rightarrow \infty} S_\omega(\chi_k) = S_\omega^{min} < 0. \quad (\text{A.4})$$

The argument of the proof of Lemma 11 guarantees that there exists a constant $C > 0$, so that

$$\|\chi_k\|_{H^1} \leq C. \quad (\text{A.5})$$

One can find $\chi_*(x) \in H^1$ so that (after taking a subsequence) χ_k tends weakly in H^1 to χ_* . Using the inequality

$$\int_{|x| > R} \frac{|\chi(x)|^2}{|x|} dx \leq \frac{C}{R},$$

and the compactness of the embedding $L^p(|x| < R) \hookrightarrow H^1(|x| < R)$, when $2 \leq p < 6$, we see that (choosing a suitable subsequence)

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^3} \frac{|\chi_k(x)|^2}{|x|} dx = \int_{\mathbb{R}^3} \frac{|\chi_*(x)|^2}{|x|} dx. \quad (\text{A.6})$$

Then we introduce φ_k, φ_* so that

$$\Delta\varphi_k = -4\pi\chi_k^2(x), \quad \Delta\varphi_* = -4\pi\chi_*^2(x).$$

One can show that φ_k tends weakly to φ_* in \dot{H}^1 . We have also the identities

$$A(\chi_k^2) = \int \varphi_k(x)\chi_k^2(x)dx = \frac{1}{4\pi}\|\nabla\varphi_k\|_{L^2}^2$$

and

$$A(\chi_*^2) = \int \varphi_*(x)\chi_*^2(x)dx = \frac{1}{4\pi}\|\nabla\varphi_*\|_{L^2}^2$$

so we obtain

$$S_\omega(\chi_k) = \frac{1}{2}\|\nabla\chi_k\|_{L^2}^2 + \frac{\omega}{2}\|\chi_k\|_{L^2}^2 + \frac{1}{4}A(\chi_k^2) - \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi_k(x)|^2}{|x|} dx$$

$$= \frac{1}{2} \|\nabla \chi_k\|_{L^2}^2 + \frac{\omega}{2} \|\chi_k\|_{L^2}^2 + \frac{1}{16\pi} \|\nabla \varphi_k\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi_k(x)|^2}{|x|} dx.$$

Using (A.4) and (A.6), we get

$$\begin{aligned} & \lim_{k \rightarrow \infty} S_\omega(\chi_k) + \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi_k(x)|^2}{|x|} dx \\ &= \lim_{k \rightarrow \infty} \frac{1}{2} \|\nabla \chi_k\|_{L^2}^2 + \frac{\omega}{2} \|\chi_k\|_{L^2}^2 + \frac{1}{16\pi} \|\nabla \varphi_k\|_{L^2}^2 = S_\omega^{min} + \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi_*(x)|^2}{|x|} dx. \end{aligned}$$

It is well - known that for any sequence f_k in a Hilbert space H tending weakly (in H) to $f_* \in H$, one has

$$\liminf_{k \rightarrow \infty} \|f_k\|_H \geq \|f_*\|_H \quad (\text{A.7})$$

and

$$\lim_{k \rightarrow \infty} \|f_k - f_*\|_H = 0 \iff \lim_{k \rightarrow \infty} \|f_k\|_H = \|f_*\|_H. \quad (\text{A.8})$$

From (A.7) we have

$$S_\omega^{min} + \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\chi_*(x)|^2}{|x|} dx \geq \|\nabla \chi_*\|_{L^2}^2 + \frac{\omega}{2} \|\chi_*\|_{L^2}^2 + \frac{1}{16\pi} \|\nabla \varphi_*\|_{L^2}^2$$

and a strict inequality is impossible since this will contradicts the definition of S_ω^{min} . Hence

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{1}{2} \|\nabla \chi_k\|_{L^2}^2 + \frac{\omega}{2} \|\chi_k\|_{L^2}^2 + \frac{1}{16\pi} \|\nabla \varphi_k\|_{L^2}^2 = \\ &= \frac{1}{2} \|\nabla \chi_*\|_{L^2}^2 + \frac{\omega}{2} \|\chi_*\|_{L^2}^2 + \frac{1}{16\pi} \|\nabla \varphi_*\|_{L^2}^2 \end{aligned}$$

and applying (A.8) we conclude that

$$\lim_{k \rightarrow \infty} \|\chi_k - \chi_*\|_{H^1} = 0.$$

This completes the proof of the Theorem.

Appendix B. Connection between the action and energy minimization problems

Consider the minimization problem

$$S_\omega^{min} = \min\{S_\omega(\chi); \chi \in H^1\}, \quad (\text{B.1})$$

associated with the action functional $S_\omega(\chi)$ and the Lions–Cazenave minimization problem

$$I_N = \min\{\mathcal{E}(\chi); \chi \in H^1, \|\chi\|_{L^2}^2 = N\}. \quad (\text{B.2})$$

As we have seen before, for every $\omega \in (1/16, 1/4)$, there exists (at most one) solution $\chi_\omega \in H^1(\mathbb{R}^3)$, which is positive and radially symmetric, and such that

$$S_\omega(\chi_\omega) = S_\omega^{min}. \quad (\text{B.3})$$

Let us denote

$$N(\omega) = \|\chi_\omega\|_{L^2}^2. \quad (\text{B.4})$$

The above definition of the function $N(\omega)$ poses the question if

$$S_\omega^{min} = I_{N(\omega)} + \frac{\omega}{2}N(\omega).$$

For completeness, in this section we shall prove the following Lemma.

Lemma 12. *If χ_1 is a solution of (B.2) with $N = N(\omega)$, then χ_1 satisfies the equation*

$$-\Delta\chi_1(x) + \int_{\mathbb{R}^3} \frac{\chi_1^2(y)dy}{|x-y|} \chi_1(x) - \frac{\chi_1(x)}{|x|} + \omega\chi_1(x) = 0. \quad (\text{B.5})$$

and

$$S_\omega(\chi_1) = \min\{S_\omega(\chi); \chi \in H^1\}.$$

PROOF. To prove the Lemma we shall follow the idea of the proof of Corollary 8.3.8 in [3]. It is obvious, that the relation

$$S_\omega(\chi_1) = \mathcal{E}(\chi_1) + \frac{\omega}{2}N(\omega),$$

guarantees that χ_1 is a minimizer of the problem

$$\min_{\|\chi\|_{L^2}^2 = N(\omega)} S_\omega(\chi).$$

Since,

$$S_\omega(\chi_1) = \min_{\|\chi\|_{L^2}^2 = N(\omega)} S_\omega(\chi) \geq \min S_\omega(\chi) = S_\omega(\chi_\omega),$$

we can use (B.4) and see that this inequality becomes equality, so

$$S_\omega(\chi_1) = \min_{\|\chi\|_{L^2}^2 = N(\omega)} S_\omega(\chi) = \min S_\omega(\chi) = S_\omega(\chi_\omega).$$

Now, the uniqueness result of Theorem 4 implies $\chi_1 = \chi_\omega$ and completes the proof.

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